

Let $A : n \times n$ and $\chi_A(\lambda)$ be the characteristic polynomial of A , i.e. χ_A is constant multiple of $\det(\lambda I - A)$, a degree n polynomial.

Cayley-Hamilton Theorem Then $\chi_A(A) = 0$. i.e. χ_A is an *annihilator* of A .

$$e^\lambda = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n$$

We define $A^0 := I_{n \times n}$,

$$e^A \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{1}{n!} A^n,$$

$$e^{At} \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{1}{n!} A^n t^n.$$

Take e^λ as dividend, and χ_A as divisor, by division: $e^\lambda = r + \chi_A q$ for some remainder $r(\lambda)$ and quotient $q(r)$ with $\deg(r) < n$. Let $r(\lambda) = \beta_0 + \beta_1 \lambda + \cdots + \beta_{n-1} \lambda^{n-1}$ for some β_i . Suppose λ_i is a root of χ_A of multiplicity m , then $\left. \frac{d^j}{d\lambda^j} e^\lambda \right|_{\lambda=\lambda_i} = e^{\lambda_i} = \left. \frac{d^j}{d\lambda^j} r(\lambda) \right|_{\lambda=\lambda_i}$ for $j < m$ since $\chi_A q$ has factor $(\lambda - \lambda_i)$. By *Cayley-Hamilton Theorem*, $e^{At} = \beta_0 I + \beta_1 A t + \cdots + \beta_{n-1} A^{n-1} t^{n-1}$, instead of infinite sum.

1st order ODE	1st order ODE system
$y' + p(x)y = q(x)$	$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{f}(t)$
integrating factor $I(x) = e^{\int p(x) dx}$	integrating factor $\mathbf{I}(t) = e^{\int -A dt} = e^{-At}$
solution $y = I^{-1}(k + \int I q dx)$	$\mathbf{x} = \mathbf{I}^{-1}(\mathbf{k} + \int \mathbf{I} \mathbf{f} dt) = e^{At}(\mathbf{k} + \int e^{-At} \mathbf{f} dt)$

I would rather use differential operator D or Laplace transform $\mathcal{L}$ to solve 1st order systems than applying the above formula because computing e^{At} is time-consuming, and we have to deal with the multiple eigenvalue case sometimes

Instead of using operator D , Laplace transform $\mathcal{L}$ or the above formula to solve systems, we should always remember a most common way, that is, finding all linearly independent homogeneous solutions first.

Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be solutions of $\dot{\mathbf{x}} = A\mathbf{x}$. If they are linearly independent, the *Wronskian* $W(\mathbf{x}_1, \dots, \mathbf{x}_n) \stackrel{\text{def}}{=} \det(\mathbf{x}_1, \dots, \mathbf{x}_n) \neq 0$. Otherwise, $W(\mathbf{x}_1, \dots, \mathbf{x}_n) \equiv 0$.

Fundamental Matrix and e^{tA}

Recall that any matrix $\mathbf{X}(t)$ satisfying $\det \mathbf{X}(t) \neq 0$ and $\dot{\mathbf{X}} = A\mathbf{X}$ is called a *fundamental matrix* for $\dot{\mathbf{x}} = A\mathbf{x}$. then arbitrary linear combination of columns of $\mathbf{X}$ is the general *homogeneous solution* $\mathbf{x}_h = \mathbf{X} \mathbf{k}$. You may use method of *undetermined coefficients* to find a *particular solution* — let $\mathbf{x}_p = \mathbf{X}(t) \mathbf{C}(t)$ for some $\mathbf{C}$, then non-homogeneous solution to $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{f}$ is $\mathbf{x} = \underbrace{\mathbf{X} \mathbf{k}}_{\mathbf{x}_h} + \underbrace{\mathbf{X} \int \mathbf{X}^{-1} \mathbf{f} dt}_{\mathbf{x}_p} = \mathbf{X}(\mathbf{k} + \int \mathbf{X}^{-1} \mathbf{f} dt)$.

The one in text is just taking $\mathbf{X}(t) := e^{tA}$ as fundamental matrix, and so $\mathbf{X}^{-1} = (e^{tA})^{-1} = e^{-tA}$. Notice that directly computing e^{tA} is usually considered a brute force. A few skill you have learned in class:

- If $A_{n \times n}$ is diagonalizable, i.e. $\exists P_{n \times n}$ such that $A = PAP^{-1}$, where Λ is a diagonal matrix with all eigenvalues on diagonal and $P = [\mathbf{v}_1 \cdots \mathbf{v}_n]$: all eigenvectors of A in columns. Then $e^{tA} = Pe^{t\Lambda}P^{-1}$,

$$e^{t\Lambda} = \begin{pmatrix} e^{t\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{t\lambda_n} \end{pmatrix} \text{ (taking fundamental matrix } X(t) := Pe^{t\Lambda}, \text{ then } e^{tA} = XP^{-1}).$$

- If a real A has eigenvalue $\lambda = \alpha + i\beta$ and eigenvector $\mathbf{v}$, then $A\bar{\mathbf{v}} = \bar{\lambda}\bar{\mathbf{v}}$, i.e. $\bar{\lambda} = \alpha - i\beta$ is an eigenvalue and $\bar{\mathbf{v}}$ is an eigenvector. Hence, $A_{n \times n} \begin{bmatrix} \operatorname{Re} \mathbf{v} & \operatorname{Im} \mathbf{v} \end{bmatrix}_{n \times 2} = \begin{bmatrix} \operatorname{Re} \mathbf{v} & \operatorname{Im} \mathbf{v} \end{bmatrix}_{n \times 2} \begin{bmatrix} \operatorname{Re} \lambda & \operatorname{Im} \lambda \\ -\operatorname{Im} \lambda & \operatorname{Re} \lambda \end{bmatrix}_{2 \times 2} = \begin{bmatrix} \operatorname{Re} \mathbf{v} & \operatorname{Im} \mathbf{v} \end{bmatrix}_{n \times 2} \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}_{2 \times 2}$. For instance, if $n = 2$, then $A = \begin{bmatrix} \operatorname{Re} \mathbf{v} & \operatorname{Im} \mathbf{v} \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \begin{bmatrix} \operatorname{Re} \mathbf{v} & \operatorname{Im} \mathbf{v} \end{bmatrix}^{-1}$.

- If A is not diagonalizable, then $\exists P$ such that $A = PJP^{-1}$, where J consists of Jordan blocks on diagonal and P consists of Jordan basis (*general eigenvectors*) of A in columns. Then $e^{tA} =$

$$Pe^{tJ}P^{-1}, e^{tJ} = \begin{pmatrix} e^{tB_1} & & 0 \\ & \ddots & \\ 0 & & e^{tB_k} \end{pmatrix}, B_i = \begin{bmatrix} \lambda_i & & \\ & \ddots & \\ & & \lambda_i \end{bmatrix} + \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} = \lambda_i I + S,$$

$$e^{tB_i} = e^{t(\lambda_i I + S)} = e^{t\lambda_i} e^{tS} = e^{t\lambda_i} \begin{pmatrix} 1 & \frac{t}{1!} & \frac{t^2}{2!} & \cdots & \frac{t^{m-1}}{(m-1)!} \\ & 1 & \frac{t}{1!} & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2!} \\ 0 & & & \ddots & \frac{t}{1!} \\ & & & & 1 \end{pmatrix}.$$

Find linearly independent solutions by eigenvalues of A and their corresponding eigenvectors:

- If $\lambda_1, \dots, \lambda_n$ (may repeat) correspond *linearly independent eigenvectors* $\mathbf{v}_1 \cdots \mathbf{v}_n$, then $\mathbf{x}_1 = e^{\lambda_1 t} \mathbf{v}_1, \dots, \mathbf{x}_n = e^{\lambda_n t} \mathbf{v}_n$ are linearly independent solutions of $\dot{\mathbf{x}} = A\mathbf{x}$.

- If $\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$, then their corresponding eigenvectors will be of the form $\mathbf{v}_1 = \mathbf{a} + \mathbf{b}i$, and $\mathbf{v}_2 = \mathbf{a} - \mathbf{b}i$, and

$$e^{\lambda_1 t} \mathbf{v}_1 = e^{\alpha t} (\cos \beta t + i \sin \beta t) (\mathbf{a} + \mathbf{b}i) = e^{\alpha t} (\cos \beta t \mathbf{a} - \sin \beta t \mathbf{b}) + ie^{\alpha t} (\sin \beta t \mathbf{a} + \cos \beta t \mathbf{b})$$

$$e^{\lambda_2 t} \mathbf{v}_2 = e^{\alpha t} (\cos \beta t - i \sin \beta t) (\mathbf{a} - \mathbf{b}i) = e^{\alpha t} (\cos \beta t \mathbf{a} - \sin \beta t \mathbf{b}) - ie^{\alpha t} (\sin \beta t \mathbf{a} + \cos \beta t \mathbf{b}),$$

i.e. span of $\begin{pmatrix} e^{\lambda_1 t} \mathbf{v}_1 \\ e^{\lambda_2 t} \mathbf{v}_2 \end{pmatrix}$ is the same as span of $\begin{pmatrix} e^{\alpha t} (\cos \beta t \mathbf{a} - \sin \beta t \mathbf{b}) \\ e^{\alpha t} (\sin \beta t \mathbf{a} + \cos \beta t \mathbf{b}) \end{pmatrix}$

- If $A\mathbf{v} = \lambda\mathbf{v}$, $\mathbf{x}_1 = e^{\lambda t} \mathbf{v}$ is a solution of $\dot{\mathbf{x}} = A\mathbf{x}$. Let $\mathbf{x}_2 = e^{\lambda t} (t\mathbf{v} + \mathbf{w})$ be a solution independent of $\mathbf{x}_1$, then $\mathbf{w}$ solves $(A - \lambda I)\mathbf{w} = \mathbf{v}$.
- If λ is a root of χ_A of multiplicity m , but the dimension of its eigenspace $(A - \lambda I) := \dim(\ker(A - \lambda I)) = k < m$, only k eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ found, but at least we know k independent solutions $\mathbf{x}_1 = e^{\lambda t} \mathbf{v}_1, \dots, \mathbf{x}_k = e^{\lambda t} \mathbf{v}_k$. The rest $(m - k)$ independent solutions are

$$\begin{aligned} \mathbf{x}_{k+1} &= e^{\lambda t} (t\mathbf{u}_k + \mathbf{u}_{k+1}), \\ &\vdots \\ \mathbf{x}_m &= e^{\lambda t} \left(\frac{t^{m-k-1}}{(m-k-1)!} \mathbf{u}_k + \cdots + \frac{t}{1!} \mathbf{u}_{m-1} + \mathbf{u}_m \right), \end{aligned}$$

where $\mathbf{u}_{k+1}, \dots, \mathbf{u}_m$ are such that

$$\mathbf{u}_k \in \ker(A - \lambda I), \text{ i.e. } \mathbf{u}_k = c_1 \mathbf{v}_1 + \dots + c_k \mathbf{v}_k, \text{ and } \begin{cases} (\lambda I - A)\mathbf{u}_{k+1} = \mathbf{u}_k, \\ \vdots \\ (\lambda I - A)\mathbf{u}_m = \mathbf{u}_{m-1}. \end{cases}$$

Note that $c_1, \dots, c_k$ are **not** arbitrary.